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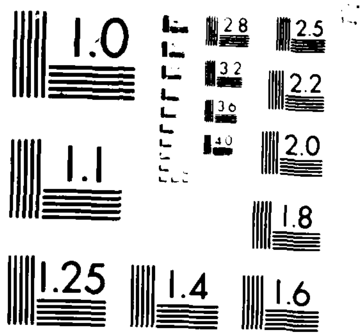

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As an application, we improve also his result on the limiting behaviour of the bilinear forms

$$B_n = \sum_{i,j=1}^{n} a_{i-j} X_i X_j,$$

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On bilinear forms in Gaussian random variables,

Toeplitz matrices and Parseval's relation

by

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On bilinear forms in Gaussian random variables, Toeplitz matrices and Parseval's relation

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Abstract. We improve a result of Szegõ on the asympototic behaviour of the trace of products of Toeplitz matrices.

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1. Statement of Results

A. We study below the asymptotic behaviour of bilinear forms

$$B_n = \sum_{i,j=1}^{n} a_{i-j} X_i X_j \tag{1.1}$$

where X_i is a mean zero stationary Gaussian sequence.

This problem was first studied in the book of Grenander and Szegõ, "Toeplitz matrices and their applications" (1958), as an application of their theory of the asymptotic behaviour of the trace of products of Toeplitz matrices.

Recently, there has been a renewed interest in this problem. See Fox and Taqqu (1983) and (1986) and Taniguchi (1986).

In Theorem 1 below we improve the results of Grenander and Szegõ on the asymptotics of the trace of products of Toeplitz matrices. This theorem can be viewed also as a generalization of Parseval's relation. As a corollary of Theorem 1, we get a result which

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improves Theorem 11.6 of Grenander and Szegö on the bilinear forms B_n (See Theorem 2).

The proof of Theorem 1 is based on a norm inequality (See Theorem 3), communicated to us by Professor Larry Brown.

In a different direction, we establish a large deviations result about B_n (See Theorem 4).

B. Let:

$$r_n = EX_0X_n \tag{1.2}$$

denote the covariance of the sequence X_n. The key fact about the bilinear form B_n is that its cumulants are:

$$\text{cum}_k(B_n) = 2^{k-1}(k-1)!Tr(A_nR_n)^k, \tag{1.3}$$

where A_n, R_n are the $n \times n$ Toeplitz matrices:

$$A_n(i,j) = a_{i-j},\ R_n(i,j) = r_{i-j}, \qquad \text{for } i,j = 1,\dots,n$$

(Formula 1.3) is an easy application of the "diagram" formula; see Rosenblatt (1985, Theorem 2.2)).

The first step in studying B_n should be thus the investigation of the asymptotic behaviour of the trace of products of Toeplitz matrices.

Let $F_n^{(\nu)}$, $\nu = 1,\dots,s$ be $n \times n$ Toeplitz matrices of the form

$$F_n^{(\nu)}(i,j) = f_{i-j}^{(\nu)} \text{ for } i,j = 1,\dots,n \quad \text{and } \nu = 1,\dots,s,$$

and suppose $f_k^{(\nu)}$ are the Fourier coefficients of the real, even functions $f^{(\nu)}(x)$, i.e.:

$$f_k^{(\nu)} = \int_{-\pi}^{\pi} e^{ikx} f^{(\nu)}(x)dx, \tag{1.4}$$

THEOREM 1. *Suppose that*

$$f^{(\nu)}(x) \in L_{p_\nu}, \qquad 1 \le p_\nu \le \infty;$$

a) *if* $\sum_{\nu=1}^{s}(p_\nu)^{-1} \le 1$, *then*

$$\lim_{n\to\infty} \frac{1}{n} Tr\Big(\prod_{\nu=1}^{s} F_n^{(\nu)}\Big) = \int_{-\pi}^{\pi} \prod_{\nu=1}^{s} (2\pi f^{(\nu)}(x))dx \tag{1.5}$$

b) *if* $\alpha > 1$, *and* $\alpha \ge \sum_{\nu=1}^{s}(p_\nu)^{-1}$, *then*

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} Tr\Big(\prod_{\nu=1}^{s} F_n^{(\nu)}\Big) = 0 \tag{1.6}$$

Remarks: 1) Formula 1.5 was first obtained by Grenander and Szezõ (1958), 7.4, under the assumption that $f^{(\nu)}(x)$ are bounded.

2) Theorem 1a is also a generalization of the classical Parseval relation. Indeed, it is shown in the Appendix that the L.H.S. of 1.5 can also be written as the Caesaro sums:

$$\frac{1}{n} Tr\Big(\prod_{\nu=1}^{s} F_n^{\nu}\Big) = \frac{A_0 + \cdots + A_{n-1}}{n}, \tag{1.7}$$

where A_k are the "skew" convolution sums:

$$A_k = \sum_{\substack{\nu_1+\cdots+\nu_s=0 \\ (\nu_1,\ldots,\nu_s)\in D_k}} f_{\nu_1} \ldots f_{\nu_s}^{(s)}, \tag{1.8}$$

and

$$D_k = \{(\nu_1, \ldots, \nu_s) : \max_{1\le j\le s} \sum_{i=1}^{j} \nu_i - \min_{1\le j\le s} \sum_{i=1}^{j} \nu_i \le k\}.$$

Thus, Theorem 1a asserts the Cesaro convergence of the "skew" convolution sums.

Note also that the "usual" convolution sums,

$$B_n = \sum_{\substack{j\in\{1,\dots,n\}^s \\ j_1+\cdots+j_s=0}} f^{(1)}_{j_1}\dots f^{(s)}_{j_s}$$

converge to the R.H.S. of 1.5, if $f^{(\nu)}(x) \in L_{p_\nu}$, $\sum_{\nu=1}^{s}(p_\nu)^{-1} \le 1$, and $1 < p_\nu < \infty$. since then the Fourier sums of $f^{(\nu)}(x)$ converge in L_{p_ν} sense, and the scalar product is continuous. In this case, taking Cesaro sums is unnecessary. If, however, some p_ν equal 1 or ∞, and $\sum_{\nu=1}^{s}(p_\nu)^{-1} = 1$, we do not know whether B_n converge to the R.H.S. of (1.5) in Cesaro sense. However, for $n = 2$ and 3, $C_n = D_n$, and in the case $n = 2$ we have the classical Parseval relation (See Katznelson, (1968), pg. 35).

As an immediate corollary of Theorem 1 we get:

THEOREM 2. *Let a_k and r_k in 1.1) and 1.2) be the Fourier coefficients of the real even functions $a(x)$ and $r(x)$, and suppose $a(x) \in L_{p_1}$, $r(x) \in L_{p_2}$, $1 \le p_1,\ p_2 \le \infty$ and*

$$(p_1)^{-1} + (p_2)^{-1} \le 2^{-1}. \tag{1.9}$$

Then,

$$\frac{B_n - E(B_n)}{\sqrt{n}} \xrightarrow{d} N(0,\sigma^2), \tag{1.10}$$

where

$$\sigma^2 = 2(2\pi)^4 \int_{-\pi}^{\pi} a^2(x) r^2(x)\,dx.$$

Proof: Use the method of cumulants:

$$\mathrm{cum}_k\Big(\frac{B_n - EB_n}{\sqrt{n}}\Big) = \begin{cases} 0 & \text{for } k=1 \\ 2\frac{Tr(A_nR_n)^2}{n} & \text{for } k=2 \\ 2^{k-1}(k-1)!\frac{Tr(A_nR_n)^k}{n^{k/2}} & \text{for } k\ge 3 \end{cases}$$

$$\underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{for } k=1 \\ 2\cdot(2\pi)^4\int_{-\pi}^{\pi} a^2(x)r^2(x)dx & \text{for } k=2, \text{ by Theorem 1a} \\ 0 & \text{for } k\ge 3, \text{ By Theorem 1b} \end{cases}$$

Notes: 1) 1.10 was first established by Grenander and Szegõ (1958, Thm. 11.6), under the assumption that $a(x)$ and $r(x)$ are bounded.

2) Taqqu and Fox (1983) extended the result of Grenander and Szegõ under a set of assumption different from ours. They show that if $a(x)$ and $r(x)$ are continuous, except maybe at 0, and are regularly varying at 0, then $a(x)r(x) \in L_2$ (which is a weaker assumption than 1.9) is sufficient for 1.10 to hold.

C. Theorem 1 follows from the following inequality, communicated to us by Larry Brown:

THEOREM 3. *For* $1 \leq p \leq \infty$,

$$\|F_n\|_p \leq n^{1/p} \|f(x)\|_p, \tag{1.11}$$

where $\|F_n\|_p = \left(\sum_{j=1}^n |s_{j,n}|^p\right)^{1/p}$, $s_{j,n}$ *being the singular values of the matrix* F_n.

(1.11) can be first established for $p = 2, \infty$ and 1. By the Riesz convexity theorem, it follows then that it holds for every p.

D. We see from Theorem 1a that when $a(x)$ and $r(x)$ are bounded, the cumulants of B_n increase all at the same asymptotic rate ($\text{cum}_k(B_n) = O(n)$). In such cases, large deviations results hold. We get, by applying Lemma 1 of Cox and Griffeath (1985), the following:

THEOREM 4. *Suppose* $a(x)$ *and* $r(x)$ *are even, real functions, which are Riemann integrable. Let* $L = 4\pi \sup_x a(x) \cdot \sup_x r(x)$, *and* $\varphi(s) = -\frac{1}{2}\int_{-\pi}^{\pi} \ell n(1 - 4\pi s a(x) r(x))dx$, *for any* $s \in (-\infty, L^{-1})$. *Then,*

a) for any $\alpha \in (\varphi'(0), \lim_{s \nearrow L^{-1}} \varphi'(s))$

$$\lim_{n \to \infty} \frac{1}{n} Pr\left\{\frac{B_n}{n} > \alpha\right\} = -I(\alpha)$$

b) for any $\alpha \in (\lim_{s \to -\infty} \varphi'(s), \varphi'(0))$

$$\lim_{n \to \infty} \frac{1}{n} Pr\left\{\frac{B_n}{n} < \alpha\right\} = -I(\alpha),$$

where $I(\alpha) = \alpha s_\alpha - \varphi(s_\alpha)$, and s_α is the unique solution of $\varphi'(s_\alpha) = \alpha$.

2. Proofs

Proof of Theorem 1: a) Let m be the number of $f^{(\nu)}$ which are non-polynomials (have infinitely many non zero Fourier coefficients). We will use induction on m. For $m = 0$ (i.e. all $f^{(\nu)}(x)$ are polynomials), it is easy to check that (1.5) holds. Suppose now (1.5) holds whenever we have at most m non-polynomials.

Consider then any set of $f^{(\nu)}(x)$ which has at most $m+1$ non-polynomials, and suppose w.l.o.g. that $f^{(1)}(x)$ is a non-polynomial. Let then $f_k^{(1)}(x)$ denote the kth Fejer sum of $f^{(1)}(x)$, let $f^{(1),k}(x) = f^{(1)}(x) - f_k^{(1)}(x)$, and let $F_{n,k}^{(1)}$ and $F_n^{(1),k}$ be the corresponding Toeplitz matrices. Then

$$\lim_{n\to\infty} \frac{1}{n} Tr(F_{n,k}^{(1)} \prod_{\nu=2}^{s} F_n^{(\nu)}) = \int_{-\pi}^{\pi} 2\pi f_k^{(1)}(x) \prod_{\nu=2}^{s} (2\pi f^{(\nu)}(x))dx \tag{2.1}$$

by the induction hypothesis, and the R.H.S. of (2.1) converges as $k \to \infty$ to $\int_{-\pi}^{\pi} \prod_{\nu=1}^{s} (2\pi f^{(\nu)}(x))dx$ since $1 \le p_1 \le \infty$ implies that $\|f_k^{(1)} - f^{(1)}\|_{p_1} \underset{k\to\infty}{\longrightarrow} 0$, and $\prod_{\nu=2}^{s} f^{(\nu)}(x) \in L_{q_1}$, where $(p_1)^{-1} + (q_1)^{-1} \le 1$. To show then that (1.5) holds with up to $m+1$ non-polynomials it remains only to note that:

$$\begin{aligned}
&\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} |Tr F_n^{(1),k} \prod_{\nu=2}^{s} F_n^{(\nu)}| \\
&\le \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \|F_n^{(1),k} \prod_{\nu=2}^{s} F_n^{(\nu)}\|_1 \\
&\le \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \|F_n^{(1),k}\|_{p_1} \prod_{\nu=2}^{s} \|F_n^{(\nu)}\|_{p_\nu} \\
&\le \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{n^{\sum_\nu (p_\nu)^{-1}}}{n} \|f^{(1),k}(x)\|_{p_1} \prod_{\nu=2}^{s} \|f^{(\nu)}(x)\|_{p_\nu} \qquad \text{(by Thm 3)} \\
&= 0.
\end{aligned}$$

□

b) Assume first w.l.o.g. $\sum_{\nu=1}^{s} (p_\nu)^{-1} > 1$. (Otherwise the result follows from a)).

The proof is now similar with that of part a). If all $f^{\nu)}(x)$ are polynomials, the limit is 0 since $\alpha > 1$. Otherwise, if say, $f^{(1)}(x)$ is a nonpolynomial, replace $f^{(1)}$ by $f^{(1),k} - f_k^{(1)}$. Finally, let $\theta = \sum_{\nu=1}^{s}(p_\nu)^{-1}$, and note that

$$\begin{aligned}
\frac{1}{n^\alpha}|Tr\ F_n^{(1),k}\prod_{\nu=2}^{s}F_n^{(\nu)}| &\leq \frac{1}{n^\alpha}\|F_n^{(1),k}\|_{\theta p_1}\prod_{\nu=2}^{s}\|F_n^{(\nu)}\|_{\theta p_\nu} \\
&\leq \frac{1}{n^\alpha}\|F_n^{(1),k}\|_{p_1}\prod_{\nu=2}^{s}\|F_n^{(\nu)}\|_{p_\nu} \qquad \text{(since } \theta > 1) \\
&\leq \frac{1}{n^\alpha}\cdot n^{\sum_{\nu=1}^{s}(p_\nu)^{-1}}\|f^{(1),k}\|_{p_1}\prod_{\nu=2}^{s}\|f^{(\nu)}\|_{p_\nu} \qquad \text{(by Theorem 3)} \\
&\leq \|f^{(1),k}\|_{p_1}\prod_{\nu=2}^{s}\|f^{(\nu)}\|_{p_\nu} \underset{k\to\infty}{\longrightarrow} 0. \qquad \square
\end{aligned}$$

Proof of Theorem 4: This is a straightforward application of Lemma 1 of Cox and Griffeath (1985). We need only to check that for any $s \in (-\infty, L^{-1})$, the cumulant generating function:

$\varphi_n(s) = \log\ Ee^{sB_n}$ satisfies:

$$\lim_{n\to\infty}\varphi_n(s) = -\frac{1}{2}\int_{-\pi}^{\pi}\ell n(1 - 4\pi s a(x)r(x))dx. \tag{2.2}$$

But $\varphi_n(s)$ equals:

$$\varphi_n(s) = -\frac{1}{2}\sum_{i=1}^{n}\ell n(1 - 2s\lambda_{i,n}),$$

where $\lambda_{i,n}$ are the eigenvalues of A_n, R_n, for any $s \leq [\underset{i}{Max}\ 2\lambda_{i,n}]^{-1}$ (Direct computation).

(2.2) follows now Theorem 4.4 ii of Gray(1971), since $a(x)$, $r(x)$ are Riemann integrable, and the function $\ell n(1 - 4\pi sz)$ is continuous for $z \in (-\infty, \frac{L}{4\pi})$, if $s < L^{-1}$. $\square$

Note: The assumption of Riemannian integrability is probably too strong. We follow however Gray in adopting it, due to the conceptual simplicity which it brings to the problem. (Under this assumption, the Toeplitz matrices are asymptotically equivalent

with circulant approximands, which are much easier to manipulate. This approach is nicely illustrated in Gray (1971)).

Acknowledgement: We thank Professor Larry Brown for communicating to us Theorem 3.

Appendix

Proof of formula 1.7: Let

$$C_n = \{1, \dots, n\}^s,$$

let $T(j_1, \dots, j_s)$ denote the range of sums $\sum_{\nu=1}^{k} j_\nu$, i.e.

$$T(j_1, \dots, j_s) = \operatorname*{Max}_{1 \le k \le s} \sum_{\nu=1}^{k} j_\nu - \operatorname*{Min}_{1 \le k \le s} \sum_{\nu=1}^{k} j_\nu,$$

let

$$D_n = \{(j_1, \dots, j_s) : \sum_{\nu=1}^{s} j_\nu = 0, T(j_1, \dots, j_s) \le n\},$$

and let A_n be the "skew" convolution sums:

$$A_n = \sum_{\underline{j} \in D_n} f_{j_1}^{(1)} \dots f_{j_s}^{(s)}.$$

Then,

$$\begin{aligned}
\frac{1}{n} Tr\left(\prod_{\nu=1}^{s} F_n^{(\nu)}\right) &= \frac{1}{n} \sum_{\underline{i} \in C_n} f_{i_1 - i_2}^{(1)} f_{i_2 - i_3}^{(2)} \dots f_{i_s - i_1}^{(s)} \\
&= \frac{1}{n} \sum_{\underline{j} \in D_{n-1}} f_{j_1}^{(1)} \dots f_{j_s}^{(s)} \sum_{\substack{i_1 - i_2 = j_1 \\ i_s - i_1 = j_s}} 1 \qquad \text{(A.1)} \\
&= \frac{1}{n} \sum_{\underline{j} \in D_{n-1}} f_{j_1}^{(1)} \dots f_{j_s}^{(s)} (n - T(j_1, \dots, j_s)).
\end{aligned}$$

The last equality holds since the set of all $\underline{i}$'s with given $\underline{j}$ differences can be obtained from any of its elements $\underline{i}^{(0)}$, by adding or subtracting $(1, \dots, 1)$ as long as all components

are in the range $\{1,\ldots,n\}$; as such, it has $(n-\max_{\nu} i_{\nu}^{(0)}) + \min_{\nu} i_{\nu}^{(0)}$ elements. Furthermore,

$$\begin{aligned}
\max_{\nu} i_{\nu}^{(0)} - \min_{\nu} i_{\nu}^{(0)} &= \max_{\nu}(-i_{\nu}^{(0)}) - \min_{\nu}(-i_{\nu}^{(0)}) \\
&= \max_{\nu}(i_1^{(0)} - i_{\nu}^{(0)}) - \min_{\nu}(i_1^{(0)} - i_{\nu}^{(0)}) \\
&= \max_{\nu}(\sum_{k=1}^{\nu} j_k) - \min_{\nu}(\sum_{k=1}^{\nu} j_k) \\
&= T(j_1,\ldots,j_s).
\end{aligned}$$

Finally, from (A.1) we get

$$\frac{1}{n} Tr(\prod_{\nu=1}^{s} F_n^{\nu)}) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{\underline{j}\in D_k} f_{j_1}^{(1)}\ldots f_{j_s}^{(s)} = \frac{1}{n}\sum_{k=0}^{n-1} A_k .$$

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